Simple forms for equations of rays in gradient-index lenses

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The calculation of the shape of an optical ray in a gradient-index lens is often greatly simplified by the use of formalism in which the equation governing the ray assumes the form of Newton's law of motion. A new justification of this method is presented, and the method is applied to a gradient-index lens with cylindrical symmetry.

I. INTRODUCTION

An article published recently in this Journal[1] discussed an especially simple version of the optical–mechanical analogy: The equation governing a ray of light in a medium of varying index of refraction can be cast into the form of Newton's law of motion.

This formulation of geometrical optics has both theoretical and practical advantages. On the theoretical side, it presents the optical–mechanical analogy in a much simpler form than usual. (The traditional expression of the analogy relies upon the Hamilton–Jacobi equation and a rather esoteric formulation of the laws of mechanics.) From the practical point of view, the "$F = ma$" formulation of geometrical optics offers substantial calculational advantages. Many problems involving gradient-index media are solved more easily with this approach than with any other.

The present article serves two purposes: (1) It offers a new proof of this formulation of geometrical optics. The new proof relies less on calculation and more on physical argument and, it is to be hoped, makes the physical basis of the analogy clearer. (2) The formalism of "$F = ma$" optics is applied to examples somewhat more complex than those treated in the earlier article. In particular, one example treats a case in which the ray is three-dimensional, i.e., not confined to a plane. The facility with which the equations governing the ray may be obtained and solved is a strong argument for the utility of this method of calculation.

II. THE FORMALISM

Many problems in gradient-index optics can be solved most easily by means of a formalism in which the equation governing the optical ray assumes the form of Newton's second law:[2]

$$\frac{d^2\mathbf{x}}{da^2} = \nabla\left(\frac{1}{2}n^2\right). \qquad (1)$$

The position $\mathbf{x}$ of a light pulse moves along a ray through a region of varying index of refraction $n(\mathbf{x})$. The equation governing the motion of the light along the ray takes the form of Eq. (1) when we use as independent variable, not the time t, but a stepping parameter a that is defined by the relation

$$\left|\frac{d\mathbf{x}}{da}\right| = n. \qquad (2)$$

In Eq. (1), the optical analog of the potential energy is $-\frac{1}{2}n^2$ and the optical analog of the mass is the number 1. With the identifications

$$t \to a,$$
$$m \to 1,$$
$$\mathbf{x}(t) \to \mathbf{x}(a),$$
$$U(\mathbf{x}) \to -\tfrac{1}{2}n^2(\mathbf{x}),$$

Eq. (1) is formally equivalent to Newton's law of motion,

$d^2\mathbf{x}/dt^2 = -\nabla U(\mathbf{x})$, and may be solved by exactly the same techniques.

In the great majority of optics problems, we desire only the *shape* of the ray. In such problems, the independent variable (a in our formalism, corresponding to t in a mechanics problem) is ultimately eliminated from the solution. Equation (2) always will suffice for this purpose. Occasionally, however, it may be necessary to find the progress in time of light along the ray. In such, comparatively rare, cases, an alternative form of Eq. (2) will permit us to pass over from the stepping parameter a to the time t as independent variable. We begin by writing Eq. (2) in the form $|d\mathbf{x}/dt|(dt/da) = n$. Then, since $|d\mathbf{x}/dt| = c/n$ (where c is the speed of light in vacuum), we obtain

$$da = (c/n^2)dt, \tag{3}$$

the explicit connection between the stepping parameter a and the time t.

III. PHYSICAL BASIS OF THE ANALOGY

Equation (1) is the basis for a simple and general approach to solving gradient-index problems in geometrical optics. All of the standard techniques of Newtonian mechanics are immediately transferable to the domain of optics. Justification of this formulation of geometrical optics is best sought in a derivation of Eq. (1), which may be found in Ref. 1. However, it may be helpful also to offer a more physical justification that makes the underlying principles evident and that provides an interpretation of the stepping parameter.

We begin with the analogy between two variational principles, the principle of least time and the principle of least action. The first of these governs the shape of a ray in a medium of variable index of refraction; and the second, the trajectory of a material particle in a time-independent, conservative force field. On the top line of Table I, these two principles are stated in their most familiar forms. In each case δ represents a variation of the integral produced by a variation in the path of integration between two fixed points in space. $|d\mathbf{x}|$ is an element of the path of integration.

In the case of Fermat's principle, the speed v of light is a function of position. The value of the integral (the total time of travel) therefore depends upon the path, and Fermat's principle states that the actual shape of the ray is the one that minimizes the time.[3]

In the case of Maupertuis' principle, the speed v of the particle is likewise to be considered a function of position.[4] Thus the rules governing the variations in the two cases are strictly analogous. The integral $\int dA \equiv \int v|d\mathbf{x}| = \int v^2\,dt$ is called the "action.[5]" It is clear that the action in mechanics plays a role analogous to that of the time in geometrical optics.

Let us define an optical quantity of the same form as an action, i.e.,

$$dA \equiv v|d\mathbf{x}| = (c/n)|d\mathbf{x}|, \tag{4}$$

for light in a medium of refractive index n. By using this relation to eliminate the element of arc $|d\mathbf{x}|$, we may cast Fermat's principle, $\delta \int v^{-1}|d\mathbf{x}| = 0$, in the form

$$\delta \int n^2\,dA = 0. \tag{5}$$

Equation (5) appears in Table I as a second form of Fermat's principle. Comparing the second forms of the two principles (lower line of Table I), we note the following correspondences:

Optics		Mechanics
$\delta \int n^2\,dA = 0$	$\to$	$\delta \int v^2\,dt = 0,$
A	$\to$	$t,$
$n(A)$	$\to$	$v(t).$

That is, the variational principle governing geometrical optics will take on the form of Maupertuis' principle if we think of A (rather than t) as independent variable and if we let the index of refraction n play the role of a particle's speed.

It remains only to demonstrate that n actually does play the role of a speed when A is taken to be the independent variable. But this is already implicit in the definition of A [Eq. (4)], for we have immediately

$$n = c\left|\frac{d\mathbf{x}}{dA}\right|.$$

Thus n actually is entitled to play the role of a speed: It is essentially the magnitude of the derivative of a position vector with respect to the independent variable.

The *stepping parameter* a used throughout this article and Ref. 1, and defined by Eq. (2) or (3), differs from A only by a constant factor c (which it would be useless to write continually). The stepping parameter a may therefore be thought of as an optical action. Its use as independent variable casts geometrical optics into the form of one-particle mechanics. The master equation (1) follows immediately in the same way that Newton's law of motion may be derived from Maupertuis' principle.[6]

Table II provides a visual image of the relationship between mechanics and the formulation of geometrical optics under consideration here. The statement made in the last sentence of the fourth paragraph of this section is obvious and familiar. But it is now clear that this statement may

Table I. Two variational principles.

Geometrical optics: Fermat's principle or principle of least time	One-particle mechanics: Maupertuis' principle or principle of least action
$\delta \int \frac{1}{v}|d\mathbf{x}| = 0$ or $\delta \int n^2\,dA = 0$	$\delta \int v|d\mathbf{x}| = 0$ or $\delta \int v^2\,dt = 0$

Table II. A schematic view of the relationship between one-particle mechanics and this article's formulation of geometrical optics.

	Optics	Mechanics
Independent variable	A	t
Minimized quantity	t	A

also be reversed: The action in optics plays a role analogous to that of the time in mechanics.

The "$F=ma$" formulation of geometrical optics exploits the formal similarity of the principles of Maupertuis and Fermat. Substantial calculational advantages are thereby gained for geometrical optics. The analogy between geometrical optics and mechanics is not, however, merely a result of mathematical manipulation. Rather, the roots of the optical–mechanical analogy are to be found in the wave nature of material particles. Indeed, the two variational principles are really but one, for Maupertuis' principle may be derived by applying Fermat's principle to de Broglie's matter waves.[7]

IV. A GRADIENT-INDEX LENS WITH CYLINDRICAL SYMMETRY

In a recent article in this Journal, Jones *et al.*[8] discussed an interesting demonstration involving the shape of a ray in a gradient-index lens with cylindrical symmetry (see Fig. 1). Choose cylindrical coordinates r, θ, and z, and let the z axis coincide with the axis of the cylinder. The index of refraction of the device considered by Jones *et al.* is of the form

$$n(r) = n_0(1 - \tfrac{1}{2}Br^2), \tag{6}$$

where n_0 and B are constants.

This cylindrical lens provides excellent opportunities for demonstrating the use of Eq. (1). The lens is complex enough to have interesting, and even surprising, properties, but still simple enough to admit of neat solutions, at least for a number of special cases. Sections V and VI, immediately below, explore the properties of this remarkable lens.

V. PROBLEM 1: APPLICATION TO A RAY IN A PLANE CONTAINING THE AXIS OF THE LENS

Consider a ray lying in a plane containing the axis of the cylinder, as shown in Fig. 1. There are a number of traditional ways to obtain a differential equation for $r(z)$, the shape of the ray. For example, the calculus of variations may be applied directly to Fermat's principle. This has been done by Jones *et al.*, with the following result:

$$r'' + Br(1 + r'^2)/(1 - \tfrac{1}{2}Br^2) = 0, \tag{7}$$

where the prime denotes differentiation with respect to z. This is a rather formidable equation. We shall see that the use of Eq. (1) not only leads to a much simpler differential equation, but also provides considerable physical insight into the behavior of the ray.

A. The equations governing the ray

The form of $n(r)$ for this cylindrical lens is given by Eq. (6). Thus the z and r components of the master equation [Eq. (1)] are

$$\frac{d^2r}{da^2} = -n_0^2Br(1 - \tfrac{1}{2}Br^2), \tag{8}$$

$$\frac{d^2z}{da^2} = 0. \tag{9}$$

From (9) it follows that dz/da is constant. This is the optical analog of conservation of momentum in a direction in which the gradient of the potential energy has a zero projection. Thus, for a given ray, we may put $dz/da = p_z$, a constant. Then $d^2r/da^2 = (p_z^2)d^2r/dz^2$, and Eq. (8) becomes

$$r'' = (-n_0^2B/p_z^2)r(1 - \tfrac{1}{2}Br^2), \tag{10}$$

where we follow Jones *et al.* in the use of the prime to denote differentiation with respect to z. This equation is equivalent to the Eq. (7) derived by Jones *et al.*, but is much simpler.

As the equivalence of (7) and (10) is not obvious, it is worthwhile to demonstrate this equivalence directly. In Fig. 2, $d\mathbf{x}$ is an infinitesimal element of the ray and is resolvable into components dr and dz. Thus

$$p_z \equiv \frac{dz}{da} = \left|\frac{d\mathbf{x}}{da}\right| \frac{dz}{(dz^2 + dr^2)^{1/2}}.$$

But $|d\mathbf{x}/da| = n$, by the definition (2) of a. Thus, using the particular form (6) assumed for $n(r)$, we get

$$p_z = n_0(1 - \tfrac{1}{2}Br^2)/(1 + r'^2)^{1/2} = \text{const.} \tag{11}$$

Equation (11) may be substituted into (10) to obtain the Eq. (7) derived by Jones *et al.* Thus (7) and (10), which look very different, really are equivalent.

B. Qualitative features of the ray

Jones *et al.* point out that Eq. (7) reduces to the form of a one-dimensional harmonic oscillator equation (with z playing the role of the time) if one suppresses terms of order B^2 and higher (which is equivalent to assuming $Br^2 \ll 1$) *and* if one supposes that $r'^2 \ll 1$, so that any part of the ray makes but a small angle with the axis. In this limit, the ray is a sinusoid. In particular, for a ray that enters the lens perpendicular to the flat face at a distance R from the axis, as in Fig. 1, $r(z) \approx R\cos(kz)$, where $k = B^{1/2}$. However, two comments are in order. First, Eq. (10) shows that it is not necessary separately to require $r'^2 \ll 1$: Equation

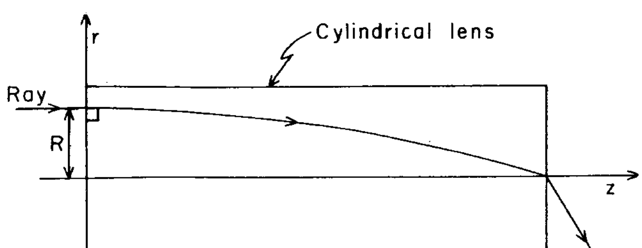

Fig. 1. An optical ray in a gradient-index lens with cylindrical symmetry.

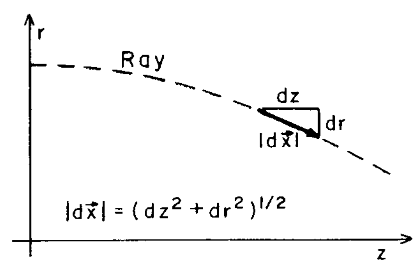

Fig. 2. An element of the ray resolved into orthogonal components.

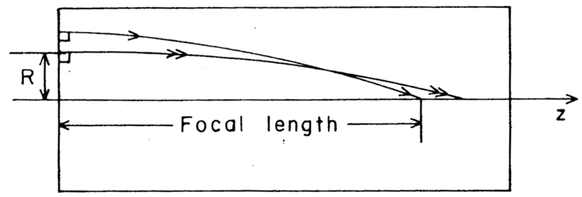

Fig. 3. An aberration of the lens: Parallel rays that enter the flat face of the lens at different values of R cross the z axis at different places.

(10) reduces to a harmonic oscillator equation as long as $Br^2 \ll 1$.

Second, if one puts $r'^2 \ll 1$, one throws away information about the focal length of the lens. That is, one loses information about the wave number k governing the sinusoidal shape of the ray. If in Eq. (10) we ignore Br^2 as being much less than unity, we obtain a sinusoidal solution; however, the wave number is not $B^{1/2}$ but $(n_0/p_z)B^{1/2}$. It is now necessary to evaluate the constant of the motion $p_z \equiv dz/da$. Let us consider the case of a normally incident ray, as in Fig. 1. At $z=0$, $r=R$ and $r'=0$, so that Eq. (11), evaluated at $z=0$, yields $p_z = n_0(1-\frac{1}{2}BR^2)$. The form of the ray inside the lens thus becomes

$$r(z) = R\cos(k_0 z),$$

where

$$k_0 = B^{1/2}(1-\tfrac{1}{2}BR^2)^{-1}.$$

It is true that k_0 reduces to $B^{1/2}$ if we impose our condition $BR^2 \ll 1$. However, this simple analysis shows that, even if we ignore departures of the ray from sinusoidal form, the wave number depends upon the radius at which the ray enters the lens. The outside part of the lens has a shorter focal length than the central part: The lens thus suffers from something analogous to spherical aberration (see Fig. 3). This property of the lens is revealed by only a cursory examination of Eq. (10), which thus presents definite advantages even in the roughest investigation of the properties of the lens.

C. Solution of the equation for $r(z)$

When one attempts actually to solve the equations, the advantages of Eq. (10) are even more considerable. We write Eq. (10) as

$$r''(z) = -k_0^2 r + \epsilon r^3, \tag{12}$$

where, for rays incident normally on the flat face at radius R,

$$k_0^2 = B(1-\tfrac{1}{2}BR^2)^{-2},$$

$$\epsilon = \tfrac{1}{2}B^2(1-\tfrac{1}{2}BR^2)^{-2}.$$

Given an equation of the form $d^2r/dz^2 = f(r)$, $r(z)$ may always be obtained by a twofold numerical integration, entirely analogous to direct numerical integration of Newton's law of motion when the potential depends only on the position. However, it is also easy to develop an analytical solution of Eq. (12) by regarding the term ϵr^3 as a perturbation of the harmonic oscillator equation. The technique is fairly straightforward and is described in standard mechanics texts that discuss nonlinear oscillators.[9] The result, to first order in the expansion parameter ϵ, is[10]

$$r(z) = R(1+BR^2/64)\cos kz - R(BR^2/64)\cos 3kz, \tag{13}$$

where

$$k = k_0(1-\tfrac{3}{16}BR^2),$$

and

$$k_0 = B^{1/2}(1-\tfrac{1}{2}BR^2)^{-1}.$$

Note, as already remarked, that k_0 is greater than $B^{1/2}$ by the factor $(1-\frac{1}{2}BR^2)^{-1}$, owing to the presence of this factor in the harmonic force itself [first term on right side of Eq. (12)]. The perturbing force ϵr^3 produces a partially compensating shift in the wave number represented by the factor $(1-3BR^2/16)$. The two effects together result in $k \approx B^{1/2}(1+5BR^2/16)$. Thus the focal length for a ray that enters the lens parallel to the axis at radius R is (to first order in BR^2)

$$\text{focal length} = (\pi/2B^{1/2})(1-\tfrac{5}{16}BR^2). \tag{14}$$

The aberration of the lens is thus displayed explicitly.[11]

VI. PROBLEM 2: RAYS IN THREE DIMENSIONS

The convenience of this approach to gradient-index optics is perhaps most strikingly demonstrated by the ease with which Eq. (1) may be applied to the full three-dimensional case. Let the ray enter the flat face of the lens at distance R from the center, and let the direction of the ray inside the lens at $z=0$ be arbitrary. This direction may be characterized by the two angles β and ϕ, as shown in Fig. 4.

A. The equations of motion

When $n(r)$ is given by Eq. (6), the components of the master equation (1) in the cylindrical system of coordinates are

$$\frac{d^2r}{da^2} - r\left(\frac{d\theta}{da}\right)^2 = -n_0^2Br + \frac{1}{2}n_0^2B^2r^3, \tag{15}$$

$$r\frac{d^2\theta}{da^2} + 2\frac{dr}{da}\frac{d\theta}{da} = 0, \tag{16}$$

$$\frac{d^2z}{da^2} = 0. \tag{17}$$

These equations replace Eqs. (8) and (9), which applied when the ray was confined to a single plane. Note that the

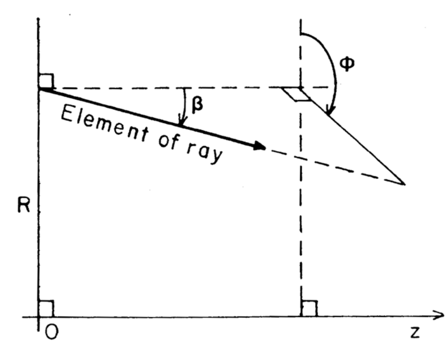

Fig. 4. The parameters R, β, and φ, which characterize the orientation of the ray inside the lens at $z=0$.

left-hand sides of Eqs. (15) and (16) are the familiar expressions for the r and θ components of the acceleration in plane polar coordinates, with t replaced by a in the derivatives. The right-hand sides of Eqs. (15)–(17) are the r, θ, and z components of the gradient of $\frac{1}{2}n^2$. This problem provides a nice—and hopefully convincing—example of the great ease with which the equations governing the ray may be written down using the formalism of Eq. (1). It is probably not unfair to say that most physicists (apart from specialists in optics) would not immediately know how to attack a problem of this sort: Gradient-index optics is not a part of the basic training of most physicists today. The advantage of Eq. (1) is that it allows one immediately to write down the "equations of motion" in tractable form. Moreover, as we shall now see, the study and solution of the equations involves procedures familiar from the study of mechanics.

B. Constants of the motion

From Eq. (17) it follows, as before, that

$$p_z \equiv \frac{dz}{da} \tag{18}$$

is a constant of the motion. From Eq. (16) it follows that

$$L_z \equiv r^2 \frac{d\theta}{da} \tag{19}$$

is also a constant of the motion. L_z is the optical analog of an angular momentum in the z direction.

C. Equations for $r(z)$ and $\theta(z)$: Qualitative discussion

As we are interested only in the *shape* of the ray, we may use $dz = p_z da$ to pass over from a to z as independent variable in Eq. (15). The other constant of the motion (L_z) may be used to eliminate $d\theta/da$. Thus Eq. (15) reduces to an effective one-dimensional problem for $r(z)$:

$$p_z^2 \frac{d^2r}{dz^2} = -n_0^2 Br + \frac{1}{2} n_0^2 B^2 r^3 + L_z^2 r^{-3}. \tag{20}$$

In a similar spirit, dividing Eq. (19) by (18) gives

$$\frac{d\theta}{dz} = \left(\frac{L_z}{p_z}\right) r^{-2}. \tag{21}$$

The shape of the ray may be specified by the two functions $r(z)$ and $\theta(z)$. The first of these may be found by solving Eq. (20). Once $r(z)$ is found, $\theta(z)$ may be obtained by the integration of Eq. (21).

The last term on the right side of Eq. (20) is the optical analog of a centrifugal force. Thus, if we ignore the term in B^2r^3, Eq. (20) has the form of the radial equation of motion for a two-dimensional harmonic oscillator, a system whose properties are well known. In this approximation ($Br^2 \ll 1$), the ray is therefore an elliptical helix. That is, when it is viewed from one end of the lens, the ray looks like an ellipse.

The term proportional to B^2r^3 in Eq. (20) may be treated as a perturbation. The effects of this term may be anticipated: (1) The shape of the ray will depart slightly from that of an elliptical helix. Moreover, (2) the distance down the z axis associated with one radial vibration and the distance down the z axis associated with one revolution in θ will no longer be equal. Thus the "ellipse" $r(\theta)$ will precess as the ray moves down the z axis. That is, it will no longer be the case that $r(\theta + 2\pi) = r(\theta)$.

D. Exact solution of a special case

Rather than attempting a general solution of Eqs. (20) and (21), let us examine an interesting special case that may be treated exactly. We seek the conditions required to produce a ray that is a cylindrical helix (i.e., a helix on a right, circular cylinder).

First, we must express the constants of the motion p_z and L_z explicitly in terms of the parameters R, β, and φ that characterize the ray just after its entry into the lens (Fig. 4). Consider p_z, which is just the z component of $d\mathbf{x}/da$. By the definition of a, $|d\mathbf{x}/da| = n$. Thus we have $p_z = n \cos\beta$. Similarly, $L_z = (r)$ times (the component of $d\mathbf{x}/da$ in the θ direction) $= rn \sin\beta \sin\varphi$. In both cases, n is to be evaluated at $z = 0$, where $r = R$. Thus

$$p_z = n_0(1 - \tfrac{1}{2}BR^2)\cos\beta, \tag{22}$$

$$L_z = Rn_0(1 - \tfrac{1}{2}BR^2)\sin\beta\sin\varphi. \tag{23}$$

Now, if a helical ray really is possible, the ray must enter the lens so that $\varphi = \pi/2$. Also we must have everywhere $dr/dz = 0$ and $r = R$. Then, Eq. (20) becomes, with the use of (23),

$$-BR^2 + \tfrac{1}{2}B^2R^4 + (1 - \tfrac{1}{2}BR^2)^2 \sin^2\beta = 0,$$

from which we find

$$\sin^2\beta = BR^2/(1 - \tfrac{1}{2}BR^2). \tag{24}$$

This determines the angle β at which the ray must start out inside the lens for a given radius R in order that a helical ray may result. For the device considered by Jones *et al.*, $B = 0.183\ \text{mm}^{-2}$ and the radius of the cylinder is 0.9 mm. If, for a concrete example, we put $R = 0.9$ mm, then $BR^2 = 0.148$ and we find $\beta = 23.6°$.

From Eqs. (21)–(24), we obtain

$$\frac{d\theta}{dz} = \frac{1}{R}\tan\beta = \frac{B^{1/2}}{(1 - 3BR^2/2)^{1/2}}. \tag{25}$$

Thus the distance Δz along the z axis associated with one turn of the helical ray is

$$\Delta z = 2\pi B^{-1/2}(1 - 3BR^2/2)^{1/2}$$

$$\approx 2\pi B^{-1/2}(1 - 3BR^2/4).$$

In view of the small size of the device, experimental demonstration of the helical ray may be difficult.

VII. CONCLUSION

The theory of rays in a medium of variable index of refraction has applications to the atmospheres of planets and stars. More significantly, gradient-index *lenses*, which once were confined to the examples of theoretical opticians, are now commercially available and will soon become common in the laboratory. This branch of geometrical optics is therefore growing in practical importance.

The approach epitomized by Eq. (1) presents substantial advantages over more traditional formulations of gradient-index optics. Equation (1) guarantees that the equations governing the ray will nearly always come out in tractable form. Even more significantly, Eq. (1) expresses a formal analogy between geometrical optics and Newtonian dynamics. The student of mechanics who already has mastered a set of techniques for dealing with the trajectories of point particles will be able to tackle gradient-index lenses with no new investment in mathematical tools. The advantages of Eq. (1) are not, however, merely computational. The physical insight and intuition of the good student of mechanics—often connected with conserved quantities such as energy, momentum, and angular momentum—can be applied directly to geometrical optics. This is a considerable advantage, considering how few students have any physical intuition for the shape of an optical ray in a region of varying index of refraction.

The chief formal difference between gradient-index optics in the formulation of Eq. (1) and the mechanics of point particles in the Newtonian formulation is the use of the stepping parameter a in place of the time as independent variable. This is a small price to pay for the benefits of this method, for the following reasons. First, as Eq. (3) shows, a increases monotonically with t. One may therefore, in the course of algebraic and other manipulations, operate as if a were the time, without fear of making a mistake. Second, in the traditional formulation of gradient-index optics, the arc length s along the ray is often used as an independent variable: One does not get to use the time in any case, and the arc length possesses very limited computational advantages as an independent variable. Third, as mentioned above, in many optical problems, we desire only the *shape* of the ray and eventually eliminate the independent variable (whether it be t, s, or a) from the solution. This was the case in the problems treated in the present article. Finally, Eq. (3) may always be used to obtain to an explicit time description if this is desired.[12]

The best way to become convinced of the advantages offered by Eq. (1) and its approach to gradient-index optics is to try one's hand at solving a few problems with it. The goal of this article has been to convince a few readers to do just that.

[1] James Evans and Mark Rosenquist, " '$F = ma$' optics," Am. J. Phys. **54**, 876–883 (1986).

[2] Section II of this article is a summary of the contents of Ref. 1, to which the reader may refer for a derivation of Eq. (1) and examples of its use.

[3] Strictly, of course, the principles of Fermat and Maupertuis require their integrals only to be stationary, and not necessarily minima. Nevertheless, "least time" and "least action" remain standard terminology.

[4] In other words, the path is to be varied subject to conservation of energy. (The energy on the varied path must be the same as the energy on the true path.) Maupertuis' principle must therefore be regarded as logically dependent upon the principle of conservation of energy. In the optical case, however, the analog of conservation of energy need not be separately imposed, for it is satisfied automatically. The optical analog of the potential energy is $U = -\frac{1}{2}n^2$ (see the list of correspondences in Sec. II). The optical analog of the speed is $|d\mathbf{x}/da|$ (see the discussion in Sec. III). Thus the optical "kinetic energy" is $T = \frac{1}{2}|d\mathbf{x}/da|^2$, since we may simply put the mass equal to unity. But, by Eq. (2), $|d\mathbf{x}/da| = n$. The "total energy," $T + U$, is therefore zero on all rays. For a further discussion of this point and its physical consequences, see Ref. 1, pp. 878–879.

[5] We have suppressed a multiplicative constant (the mass of the particle) that is usually included in the action.

[6] For a detailed derivation of Eq. (1), see Ref. 1, pp. 876–877. The central features of this derivation may be briefly outlined. One may begin by writing Fermat's principle in the form

$$\delta \int n \left| \frac{d\mathbf{x}}{da} \right| da = \delta \int n \Big(\sum_i x_i'^2 \Big)^{1/2} da = 0,$$

where $x_i' \equiv dx_i/da$, and where the index i runs over the three Cartesian coordinates. The Euler conditions that the integral be stationary are

$$\frac{\partial n}{\partial x_j} \Big(\sum_i x_i'^2 \Big)^{1/2} - \frac{d}{da} \left[n x_j' \Big(\sum_i x_i'^2 \Big)^{-1/2} \right] = 0.$$

Then, since $(\Sigma_i x_i'^2)^{1/2} = n$, by the definition of a, we immediately obtain Eq. (1). For a derivation of Newton's law of motion from Maupertuis' principle, modeled on the original derivation by Lagrange, see Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Mechanics* (Dover, New York, 1979), pp. 29–31. To derive Eq. (1) by a parallel calculation, one should begin with Fermat's principle in the form $\delta \int n|d\mathbf{x}| = 0$. Since $n = |d\mathbf{x}/da|$, this may also be written $\delta\, \Sigma_i \int x_i'\, dx_i = 0$, where the prime denotes differentiation with respect to the stepping parameter a, and the index i runs over the three Cartesian coordinates. The steps in the calculation would then be exactly the same as those given by Yourgrau and Mandelstam in their derivation of Newton's law of motion from Maupertuis' principle, except that components of the velocity $d\mathbf{x}/dt$ are replaced by the components of $d\mathbf{x}/da$. One other difference should be noted. In the derivation of Newton's law of motion, Yourgrau and Mandelstam make explicit use of the principle of conservation of energy. In a parallel derivation of Eq. (1), no similar principle need be invoked, for its place is filled by the relation $-\frac{1}{2}n^2 + \frac{1}{2}|d\mathbf{x}/da|^2 = 0$, which is nothing more or less than the definition of a (see also Ref. 4). (To avoid any possibility of confusion, it should be pointed out that the prime is used to denote differentiation with respect to a only in this note, and not in the main body of the article.)

[7] Mark Rosenquist and James Evans, "The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation," Am. J. Phys. **56**, 881–882 (1988).

[8] Kevin M. Jones, Scott Lundgren, and Alak Chakravorty, "A calculus of variations demonstration: The gradient index lens," Am. J. Phys. **56**, 1099 (1988). The constant B that appears in Eq. (6) is called A in Jones *et al.* The symbol has been changed here to avoid the possibility of confusion with the action A introduced above.

[9] The solution of the symmetric nonlinear oscillator equation [Eq. (12)] is discussed very clearly by J. B. Marion, *Classical Dynamics of Particles and Systems* (Academic, New York, 1970), 2nd ed., pp. 167–169. (Note, however, that there are some minor algebraic errors in Marion's solution in order ϵ^2.) This material was dropped from the 3rd ed.

[10] If desired, the solution is easily extended to order ϵ^2 and higher, in which case higher odd harmonics, such as cos $5kz$, appear in the solution, but with amplitudes reduced by higher powers of BR^2.

[11] For the device considered by Jones *et al.*, $B = 0.183\ \text{mm}^{-2}$ (for light of wavelength 633 nm). Thus $(\pi/2)B^{-1/2} = 3.7$ mm, which happens also to be the physical length of the cylinder: This lens is designed to be used as a diverging lens for the collimated beam from a helium–neon laser. To a first approximation (i.e., to zeroth order in BR^2), all rays are brought to a focus on the axis at the back face of the lens, from which they then diverge. However, as radius of the device considered by Jones *et al.* is 0.9 mm, Eq. (14) reveals that the focal length for the outermost rays is 4.6% shorter than the focal length for the central rays.

[12] For examples of the use of Eq. (3) in passing over to a time description, see Ref. 1, pp. 881–882.